Brownian motion and Navier-Stokes equations

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16th Workshop on Markov Processes and Related Topics,
July 15, 2021

Navier-Stokes equations

The Navier-Stokes equation on $\mathbb{R}^n$ or on a torus $\mathbb{T}^n$,

$$\partial_t u_t + (u_t \cdot \nabla) u_t - \nu \Delta u_t + \nabla p_t = 0,\ \nabla \cdot u_t = 0,\ u|_{t=0} = u_0,$$

describes the evolution of the velocity u_t of an incompressible viscous fluid with kinematic viscosity $\nu > 0$, as well as the pressure p_t. Condition $\nabla \cdot u_t = 0$ means that u_t is of divergence free.

A very good reference is :

• R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, Second edition. CBMS-NSF Regional Conference Series in Applied Mathematics, 66, SIAM, Philadelphia, PA, 1995.

In the case of $\mathbb{R}^3$, to a velocity u_t, we associate the vorticity ξ_t, which is a vector field on $\mathbb{R}^3$ defined by

$$\xi_t = \nabla \times u_t.$$

A velocity $\begin{pmatrix} u_t^1 \\ u_t^2 \end{pmatrix}$ in $\mathbb{R}^2$ can be seen as a velocity

$u_t = \begin{pmatrix} u_t^1(x,y) \\ u_t^2(x,y) \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$ so that $\xi_t = \begin{pmatrix} 0 \\ 0 \\ \partial_x u_t^2 - \partial_y u_t^1 \end{pmatrix}$.

It is obvious that $u_t \cdot \xi_t = 0$. However in general in $\mathbb{R}^3$,

$$u_t \cdot \xi_t \neq 0.$$

The term $u_t \cdot \xi_t$ is called helicity density, a term to be handled.

When u_t is a solution to the NS equation:

$$\partial_t u_t + (u_t \cdot \nabla) u_t - \nu \Delta u_t + \nabla p_t = 0, \quad \nabla \cdot u_t = 0,$$

then the vorticity ξ_t satisfies the following heat equation

$$\frac{\partial \xi_t}{\partial t} + \nabla_{u_t} \xi_t - \nu \Delta \xi_t = \nabla^s_{\xi_t} u_t$$

where $\nabla^s u_t$ is the symmetric part of ∇u_t, sometimes called the strain tensor, also to be handled. Again in $\mathbb{R}^2$, the right hand of above equality is 0, since the components of u is dependent of x, y, while ξ_t acts on z.

Question: What relation exists between these two quantities: helicity $u_t \cdot \xi_t$ and strain tensor $\nabla^s_{\xi_t} u_t$?

Theorem

For $n = 3$, there exists a family of connection ∇^t on $\mathbb{R}^3$ such that

$$\frac{1}{2\nu^2}(u_t \cdot \xi_t)^2 - \frac{1}{\nu}\nabla^s_{\xi_t} u_t \cdot \xi_t = \widehat{Ric}^t(\xi_t) \cdot \xi_t.$$

Question: What is ∇^t and from where comes the term $\widehat{Ric}^t$?

This talk is based on a joint work with Zhongmin QIAN.

Doing inner product in L^2 on the vorticity equation yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\xi_t|^2\, dx + \nu \int_{\mathbb{R}^3} |\nabla \xi_t|^2\, dx = \int_{\mathbb{R}^3} \nabla^s_{\xi_t} u_t \cdot \xi_t\, dx,$$

since $\int \nabla_{u_t}\xi_t \cdot \xi_t\, dx = \frac{1}{2}\int \mathcal{L}_{u_t}|\xi_t|^2\, dx = 0.$

Rolling a flat Brownian motion on a manifold

More than 40 years ago, Eells, Elwothy and Malliavin constructed the Brownian motion on a Riemannian manifold M by rolling without friction a flat Brownian motion of $\mathbb{R}^n$. Let $O(M)$ be the principal bundle of orthonormal frames : an element $r \in O(M)$ is an isometry from $\mathbb{R}^n$ onto $T_{\pi(r)}M$ where $\pi : O(M) \to M$ is the projection. The Levi-Civita connection on M gives rise to n horizontal vector fields $\{A_1, \ldots, A_n\}$ on $O(M)$, which are such that $d\pi(r) \cdot A_r = r\varepsilon_i$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the canonical basis of $\mathbb{R}^n$. A vector field v on M can be lift to a horizontal vector field V on $O(M)$ such that $d\pi(r)V_r = v_{\pi(r)}$. Consider SDE on $O(M)$

$$dr_t = \sum_{k=1}^{n} A_k(r_t) \circ dW_t^k + V_t(r_t)dt, \quad r_{|_{t=0}} = r_0$$

where $W_t = (W_t^1, \cdots, W_t^n)$ is a standard Brownian motion on $\mathbb{R}^n$.

Denote by $r_t(w, r_0)$ the solution to above SDE and $x_t = \pi(r_t)$. We assume that the life-time $\zeta = +\infty$ almost surely. Then for any $f \in C_c^2(M)$,

$$\frac{d}{dt}_{|_{t=0}} P_t(f \circ \pi) = \big((\frac{1}{2}\Delta_M + \mathcal{L}_v)f\big) \circ \pi,$$

where $P_t(f \circ \pi) = \mathbb{E}(f \circ \pi(r_t))$.

In Chapter V of the book
N. Ikeda, S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland, Math. Library, 24, 1981,

Ikeda and Watanabe remarked that any diffusion having above generator can be constructed by rolling without friction the same flat Brownian motion, but changing connections, with respect to another metric compatible one Γ^v.

Let $\{B_1, \ldots, B_n\}$ be the horizontal vector fields on $O(M)$ with respect to Γ^v, consider SDE on $O(M)$:

$$dr_w(t) = \sum_{i=1}^{n} B_i(r_w(t)) \circ dW_t^i, \quad r_w(0) = r_0.$$

Ikeda and Watanabe choosed a suitable connection Γ^v such that the generator of diffusion process $t \to x_t(w) = \pi(r_w(t))$ again is $\frac{1}{2}\Delta_M + v$. In fact, these vector fields $\{B_1, \ldots, B_n\}$ are such that

$$\frac{1}{2}\sum_{j=1}^{n} \mathcal{L}_{B_j}^2(f \circ \pi) = \big((\frac{1}{2}\Delta_M + v)f\big) \circ \pi.$$

This connection Γ^v was defined locally in Ikeda-Watanabe's book.

We get the following global expression:

Theorem

Let ∇^v be the covariant derivative with respect to the connection Γ^v, and ∇^0 with respect to the Levi-Civita connection. Then for two vector fields X, Y on M,

$$\nabla^v_X Y = \nabla^0_X Y - \frac{2}{n-1} K_v(X, Y),$$

where

$$K_v(X, Y) = \langle Y, v \rangle X - \langle X, Y \rangle v.$$

We remark that ∇^v provides a more information than v: the dimension of the space.

The torsion tensor T^v associated to ∇^v:

$$T^v(X, Y) = \nabla^v_X Y - \nabla^v_Y X - [X, Y],$$

has explicit expression

$$T^v(X,Y) = \frac{-2}{n-1}\Big(\langle Y, v\rangle X - \langle X, v\rangle Y\Big).$$

Moreover, T^v is skew-symmetric (TSS), that is

$$\langle T^v(X,Y), Z\rangle = -\langle T^v(Z,Y), X\rangle$$

holds for all $X, Y, Z \in \mathcal{X}(M)$ if and only if $v = 0$. In other words, T^v is not of TSS for $v \neq 0$.

We asked ourselves what we could obtain with this connection Γ^v?

Theorem

Let Ric^0 be the Ricci curvature associated to ∇^0, and Ric^v to ∇^v. Then

$$\mathrm{Ric}^v(X) = \mathrm{Ric}^0(X) - \frac{4(n-2)}{(n-1)^2} K_v(X, v) + \frac{2(n-2)}{n-1}\nabla^0_X v + \frac{2}{n-1}\mathrm{div}(v)\, X.$$

We call intrinsic Ricci tensor associated to Γ^v the following term

$$\widehat{\mathrm{Ric}^v}(X) = \mathrm{Ric}^v(X) + \sum_{i=1}^{n} (\nabla^v_{e_i} T^v)(X, e_i).$$

This term was first introduced by Bruce Driver in

• B. Driver, A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact manifold, *J. Funct. Anal.*, 109 (1992), 272-376. It appeared in Weitzenböck formula with respect to connection satisfying TSS. For a connection, which is not of TSS, the Weitzenböck formula was studied by Elworthy, Li and LeJan in

•K.D. Elworthy, Y. Le Jan, X.M. Li, *On the geometry of diffusion operators and stochastic flows*, Lecture Notes in Mathematics, 1720, Springer-Verlag, 1999.

Theorem

Assume $\dim(M) = 3$*, then* $\widehat{\mathrm{Ric}}^v$ *admits the following simple expression:*

$$\widehat{\mathrm{Ric}}^v = \mathrm{Ric}^0 + 2v \otimes v + 2\nabla^{0,s} v,$$

where $\nabla^{0,s} v$ *denotes the symmetric part of* $\nabla^0 v$*.*

Proof. First the sum $S := \sum_{i=1}^{n} (\nabla^v_{e_i} T^v)(X, e_i)$ is equal to

$$\frac{-2}{n-1}\Big(\mathrm{div}(v)\, X - \sum_{i=1}^{n} \langle X, \nabla^0_{e_i} v\rangle\, e_i\Big) + \frac{4}{(n-1)^2}\Big((n-1)|v|^2 X - K_v(X,)\Big).$$

$n = 3$ yields

$$S = -\mathrm{div}(v)\, X + \sum_{i=1}^{3} \langle X, \nabla^0_{e_i} v\rangle\, e_i + 2|v|^2 X - K_v(X, v).$$

On the other hand, for $n = 3$,

$$\mathrm{Ric}^v(X) = \mathrm{Ric}^0(X) - K_v(X, v) + \nabla^0_X v + \mathrm{div}(v)\, X.$$

Note that

$$\sum_{i=1}^{3} \langle X, \nabla^0_{e_i} v\rangle\, e_i + \nabla^0_X v = \sum_{i=1}^{3} \Big(\langle X, \nabla^0_{e_i} v\rangle + \langle \nabla^0_X v, e_i\rangle \Big)\; e_i,$$

which is nothing but $2\nabla^{0,s}_X v$. Summing them, we get

$$\widehat{\mathrm{Ric}^v}(X) = \mathrm{Ric}^0(X) + 2|v|^2 X - 2K_v(X, v) + 2\nabla^{0,s}_X v.$$

Recall that $K_v(X, Y) = \langle Y, v\rangle\, X - \langle X, Y\rangle\, v$, then
$K_v(X, v) = |v|^2 X - \langle X, v\rangle v$. We get
$\widehat{\mathrm{Ric}^v}(X) = \mathrm{Ric}^0(X) + 2\langle X, v\rangle v + 2\nabla^{0,s}_X v$.

Navier-Stokes equations on manifolds

Let M be a complete Riemannian manifold M of dimension n, with Ricci curvature bounded from below. We are interested in the following NS on M defined with the De Rham-Hodge Laplacian $\square$,

$$\begin{cases} \partial_t u_t + \nabla_{u_t} u_t + \nu \square u_t = -\nabla p_t, \\ \mathrm{div}(u_t) = 0, \quad u|_{t=0} = u_0. \end{cases}$$

Let's first say a few words on the definition of $\square$ on vector fields. There is a one-to-one correspondence between the space of vector fields $\mathcal{X}(M)$ and that of differential 1-forms $\Lambda^1(M)$.

On a local chart U, as usual, we denote by $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ the basis of the tangent space $T_x M$ and by $\{dx^1, \ldots, dx^n\}$ the dual basis of $T_x^* M$. Set $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$ and $g^{ij} = \langle dx^i, dx^j \rangle$.

Let u be a vector field on M; on U, $u = \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}$. Then the associated differential form $\tilde{u}$ has the expression

$$\tilde{u} = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} g_{ij} u_j \Big) dx^i.$$

For a differential 1-form $\omega = \sum_{j=1}^{n} \omega_j dx^j$, the associated vector field $\omega^{\#}$ has the expression

$$\omega^{\#} = \sum_{i=1}^{n} \Big(\sum_{\ell=1}^{n} g^{i\ell} \omega_\ell \Big) \frac{\partial}{\partial x_i}.$$

We have: $(\omega, A) = \langle \omega^{\#}, A \rangle = \langle \omega, \tilde{A} \rangle$ and $\Box A = (\Box \tilde{A})^{\#}$, where $\Box = dd^* + d^* d$.

Let u be a vector field on $\mathbb{R}^3$: $u = u_1 \dfrac{\partial}{\partial x} + u_2 \dfrac{\partial}{\partial y} + u_3 \dfrac{\partial}{\partial z}$. The associated differential form $\tilde{u}$ has the expression:

$$\tilde{u} = u_1 dx + u_2 dy + u_3 dz.$$

The exterior derivative $d\tilde{u}$ is given by

$$d\tilde{u} = \Big(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\Big) dz \wedge dx + \Big(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\Big) dx \wedge dy + \Big(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\Big) dy \wedge dz.$$

Hodge star $*$ operator gives an isomorphism between Λ^2 and Λ^1:

$$*d\tilde{u} = \Big(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\Big) dx + \Big(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\Big) dy + \Big(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\Big) dz.$$

So that $*d\tilde{u}$ is the differential form associated to $\xi = \nabla \times u$.

In what follows, we assume that the dimension of M is 3.

Let u_t be a solution to the Navier-Stokes equation on M,

$$\partial_t u_t + \nabla_{u_t} u_t + \nu \Box u_t = -\nabla p_t, \quad \mathrm{div}(u_t) = 0,\ u|_{t=0} = u_0.$$

Let $\tilde{u}_t$ be the associated differential form, and we set

$$\omega_t = *d\tilde{u}_t,$$

which is differential 1- form on M. We get

$$\partial_t \omega_t + \nabla_{u_t} \omega_t + \nu \Box \omega_t = \omega_t \triangleleft (\nabla^s u_t).$$

where $\omega_t \triangleleft (\nabla^s u_t)$ is a one form such that

$$\omega_t \triangleleft (\nabla^s u_t)(X) = \omega_t(\nabla^s_X u_t).$$

Using Weitzenböck formula $\square = -\Delta + \mathrm{Ric}$, a formal computation leads to

$$\frac{1}{2}\frac{d}{dt}\int_M |\omega_t|^2\,dx + \nu\int_M |\nabla\omega_t|^2\,dx$$
$$= -\nu\int_M \langle \mathrm{Ric}^0\,\omega_t, \omega_t\rangle\,dx + \int_M \langle \omega_t \triangleleft \nabla^s u_t, \omega_t\rangle\,dx.$$

A very good reference to this topic is

M. Taylor, *Partial Differential Equations III: Nonlinear Equations*, Vol. 117, Applied Mathematical Sciences, Springer New York second edition (2011).

Diffusion processes for Vorticity

Recall that $\{A_1, \ldots, A_n\}$ on $O(M)$ are canonical horizontal vector fields on $O(M)$, with respect to Levi-Civita connection and $\pi : O(M) \to M$. Following P. Malliavin, for a differential 1-form, ω, we define

$$F^i_\omega(r) = (\omega_{\pi(r)}, r\varepsilon_i) = (\pi^*\omega, A_i)_r, \quad i = 1, \ldots, n.$$

Then we have

$$(\mathcal{L}_{A_j} F^i_\omega)(r) = (\nabla_{r\varepsilon_j}\omega, r\varepsilon_i) = (\nabla\omega, r\varepsilon_j \otimes r\varepsilon_i).$$

and

$$\Delta_{O(M)} F^i_\omega := \sum_{j=1}^n \mathcal{L}^2_{A_j} F^i_\omega = (\Delta\omega, r\varepsilon_i) = F^i_{\Delta\omega}(r).$$

Let U_t be the horizontal lift of u_t to $O(M)$, then

$$(\mathcal{L}_{U_t} F^i_\omega)(r) = \langle \nabla_{u_t}\omega, r\varepsilon_i \rangle = F^i_{\nabla_{u_t}\omega}(r).$$

Let $\phi_t = \omega_t \triangleleft \nabla^s u$; then

$$F^i_{\phi_t}(r) = \omega_t(\nabla^s_{r\varepsilon_i} u_t) = \sum_{j=1}^{n} \langle \nabla^s_{r\varepsilon_i} u_t, r\varepsilon_j \rangle F^j_{\omega_t}.$$

Define $K_{ij}(t,r) = \langle \nabla^s_{r\varepsilon_i} u_t(\pi(r)), r\varepsilon_j \rangle$ and $K(t,r) = (K_{ij}(t,r))$, then

$$F_{\phi_t}(r) = K(t,r) F_{\omega_t}(r).$$

Therefore ω_t is solution to

$$\partial_t \omega_t + \nabla_{u_t} \omega_t + \nu \Box \omega_t = \omega_t \triangleleft (\nabla^s u),$$

if and only if F_{ω_t} is solution to the following heat equation defined on $O(M)$, but taking values in flat space $\mathbb{R}^n$

$$\frac{d}{dt} F_{\omega_t} = \nu \Delta_{O(M)} F_{\omega_t} - \mathcal{L}_{U_t} F_{\omega_t} + (K(t,\cdot) - \nu \operatorname{ric}) F_{\omega_t}.$$

Now consider the following SDE on $O(M)$,

$$dr_w(t) = \sqrt{2\nu}\sum_{i=1}^{n} A_i(r_w(t)) \circ dW_t^i - U_t(r_w(t))\,dt,$$

and the resolvent equation

$$\frac{d}{dt}Q_t = Q_t J_t(r_w(t)), \quad J_t(r) = K(t,r) - \nu\,\mathrm{ric}_r$$

Then for any $T > 0$,

$$t \to M_t := Q_t \; F_{\omega_{T-t}}(r_w(t)) \quad \text{is a martingale over } [0,T].$$

Consider the following SDE on $O(M)$:

$$dr_w(t) = \sqrt{2\nu}\sum_{i=1}^{n} B_i(r_w(t)) \circ dW_t^i, \quad r_w(0) = r,$$

where $\{B_1, \ldots, B_n\}$ are associated to connection Γ^v such that

$$\frac{1}{2}\sum_{j=1}^{n}\mathcal{L}_{B_j}^2(f\circ\pi) = \big((\frac{1}{2}\Delta_M + v)f\big)\circ\pi.$$

For the second SDE, the generator is

$$\nu \sum_{i=1}^{n} \mathcal{L}_{B_i}^2 (f \circ \pi) = \Big((\nu \Delta_M + 2\nu v) f \Big) \circ \pi.$$

For the first SDE, the generator is $\Big((\nu \Delta_M - u_t) f \Big) \circ \pi$.
Therefore we take

$$v = -\frac{u_t}{2\nu}.$$

Now we denote $\widehat{\mathrm{Ric}}^t$ instead of $\widehat{\mathrm{Ric}}^{-u_t/2\nu}$. Recall that

$$\widehat{\mathrm{Ric}}^v = \mathrm{Ric}^0 + 2v \otimes v + 2\nabla^{0,s} v,$$

We have

$$\widehat{\mathrm{Ric}}^t = \mathrm{Ric}^0 + \frac{1}{2\nu^2} u_t \otimes u_t - \frac{1}{\nu} \nabla^{0,s} u_t.$$

In the case of $\mathbb{R}^3$,

$$\widehat{\mathrm{Ric}}^t(\xi_t) \cdot \xi_t = \frac{1}{2\nu^2} (u_t \cdot \xi_t)^2 - \frac{1}{\nu} \nabla^s_{\xi_t} u_t \cdot \xi_t.$$

Here are integral forms:

Theorem

Let $\dim(M) = 3$. *The vorticity* ω_t *satisfies a priori identity:*

$$\frac{1}{2}\frac{d}{dt}\int_M |\omega_t|^2\, dx + \nu \int_M |\nabla \omega_t|^2\, dx$$
$$= -\nu \int_M \langle \mathrm{Ric}^0\, \omega_t, \omega_t\rangle\, dx + \int_M \langle \omega_t \triangleleft \nabla^s u_t, \omega_t\rangle\, dx.$$

Theorem

Let $\dim(M) = 3$, *the following identity holds ,*

$$\frac{1}{2}\frac{d}{dt}\int_M |\omega_t|^2\, dx + \nu \int_M |\nabla^0 \omega_t|^2\, dx$$
$$= \frac{1}{2\nu}\int_M (\omega_t, u_t)^2\, dx - \nu \int_M (\widehat{\mathrm{Ric}^t}^{\#}\, \omega_t, \omega_t)\, dx.$$

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